AIAA JOURNAL
Vol. 39, No. 10, October 2001

Eigenstrain Without Stress and Static Shape Control of Structures

Hans Irschik*
Johannes Kepler University of Linz, A-4040 Linz, Austria
and
Franz Ziegler†
Technical University of Vienna, A-1040 Vienna, Austria

The present contribution explores two fundamental aspects of eigenstrain analysis in three-dimensional bodies. At first, distributions of eigenstrain are derived that do not cause stresses, so-called stress-free or impotent eigenstrains. We consider bodies of finite extent with geometric surface constraints, such as imposed by immovable supports or rigidly clamped boundaries. Within the setting of anisotropic linear elastic bodies, it is verified that a field of eigenstrains that is equal to the field of strains produced by external forces is a stress-free one and that the deformations caused by these eigenstrains and the deformations caused by the forces are equal. Hence, the stress-free eigenstrain load represents an exact solution for the static shape control problem of bodies acted upon by forces. Additionally, nonuniqueness of this shape control problem is demonstrated, and three-dimensional eigenstrains responsible for that nonuniqueness are identified. This is performed by showing that incompatible distributions of eigenstrain and the strains generated by these fields, when applied as a compatible distribution of eigenstrain, result in identical deformations and stresses. Deformation-free fields then result by applying the difference between those fields of eigenstrain.

Introduction

LINEAR elastic structures are being considered under quasistatic conditions with respect to their deformations. The geometrically linearized theory of elasticity is taken into account. Special emphasis is given to the analysis of the behavior of solids and structures actuated by eigenstrains. "Eigenstrain" is a generic name originally given by Mura[1] to inelastic strains resulting from thermal expansion, phase transformation, initial strains, plastic strains, and misfit strains.

Eigenstrain analysis tries to render an efficient strategy with respect to both, analytical and numerical computations, e.g., the authors of the present contribution and their coworkers have applied such eigenstrain analyses in a series of papers to static and dynamic problems of elastic-viscoplastic and damaging structures; see Refs. 2–4. Furthermore, it has been pointed out by our groups that piezoelectric strains can be treated effectively as eigenstrains; see, e.g., Refs. 5–14. It has been demonstrated in Refs. 5–14 that eigenstrain analysis can be applied directly and successfully to "intelligent," "smart," or "adaptive" structures, which utilize piezoelectricity for the sake of structural actuation, sensing, and control in an integrated circuit. An overview on technology of intelligent structures designed for aerospace applications was presented by Crawley.[15] Piezoelectricity and its application in disturbance sensing and control of flexible structures were reviewed by Rao and Sunar.[16] The latter extensive reviews contain also a large list of references. Briefly, the piezoelectric effect has been discovered by the Curie brothers in 1880. Since then, a tremendous effort has been put in the understanding and application of piezoelectric materials. The importance of the piezoelectric effect, exhibited by conventional ferroelectric polycrystals, natural crystals, and special polymers, is as a result of the conversion of mechanical into electrical energy and vice versa. In the linearized theory no body forces are produced by the applied electric field, and the constitutive relations therefore are the only source of coupling between the electric and the mechanical field. Hence, the electric field acts as an eigenstrain upon the piezoelastic body. Many practical problems, especially in the field of active vibration control, have been solved successfully by utilizing piezoelastic materials, mostly in the context of laminated piezoelectric beams, plates, and shells (see a recent review by Saravanos and Heyliger[17]).

The present contribution explores two fundamental aspects of eigenstrain analysis. First, distributions of eigenstrain are derived that do not cause stresses, so-called stress-free or impotent eigenstrains; see again Mura.[1] However, the present investigation is no longer restricted, neither to infinite nor to free bodies, yet even structures of finite extent are considered with geometric surface constraints, such as imposed by immovable supports or rigidly clamped boundaries. Within a three-dimensional setting of anisotropic bodies, it is subsequently verified that a field of eigenstrains which is equal to the field of strains produced by external forces is a stress-free one and that the deformations caused by these eigenstrains and the deformations caused by the forces are equal.

Second, this stress-free eigenstrain load is demonstrated to present an exact solution for the static shape control problem of structures acted upon by forces. Static shape control, in general, renders a complex inverse problem. In the context of the eigenstrain load, however, the static shape control problem reduces itself to the following question: what kind of spatial distribution of eigenstrain should be imposed on a structure in order to match the resulting field of deformation of the structure to a desired one? Especially the applications of static shape control to large space structures and to smart structures are topics of current interest.[18–20] With respect to smart structures, this problem has been tackled by nonlinear optimization techniques, where a finite number of actuator patches has been applied to the structure.[20,21] Because a finite number of actuator patches is used in the cited papers, a desired deflection of a flexible distributed-parameter system can be produced only approximately. In the present paper no such restriction is made; it is assumed that the proper eigenstrain load can be applied throughout the structure and without a limitation of its intensity. As is shown subsequently, the preceding inverse problems can be exactly solved under these conditions. The presented solutions are felt to provide a deep insight into the characteristic features of the deformation control. A reduction to a finite number of actuators can be easily performed, and the presented solutions can serve as a first guess in those cases where limitations of the eigenstrain load have to be taken into account. The results thus derived put our preliminary studies on static and dynamic piezoelectric actuation of beam- and plate-type structures (see again Refs. 5–14) in the context of static stress-free eigenstrains in a three-dimensional setting.

Received 17 December 1999; revision received 17 December 2000; accepted for publication 6 April 2001.

*Professor of Technical Mechanics, Abteilung für Technische Mechanik, Altenbergerstraße 69. Member AIAA.

†Professor of Rational Mechanics, Institut für Allgemeine Mechanik, Wiedner Hauptstr. 8-10/E201; franz.ziegler@tuwien.ac.at.

Additionally, nonuniqueness and hence the characterization of the shape control problem as ill-posed are demonstrated in the following, and three-dimensional fields of eigenstrains responsible for that nonuniqueness are identified. The ill-posed inverse problem has been treated by Irschik et al.[5,12] for the case of flexural vibrations of piezoelectric beams, where it has been attributed to an electric field without deflection. Nonuniqueness is overcome in the present contribution by deriving three-dimensional deformation-free fields of eigenstrain, which can be added to the present solution of the shape control problem to form another solution. Deformation-free eigenstrains are derived next by showing that incompatible distributions of eigenstrain and the strains generated by these fields, when applied as a compatible distribution of eigenstrain, result in identical deformations and stresses. Deformation-free fields then result by applying the difference between those fields of eigenstrain.

As an illustrative but still sufficiently simple example, a redundant truss with a nonuniform distribution of stiffness is studied, and Castigliano's theorem is applied in a further step for an independent proof.

Basic Equations of Eigenstrain Analysis

For an anisotropic body and in the presence of eigenstrains, the elastic strain is related to stress by the generalized Hooke's law, Reißner[22] (see again Mura[1]):

$$\varepsilon_{ij(\varepsilon)} - \bar{\varepsilon}_{ij} = C_{ijlm}\sigma_{lm(\varepsilon)}, \qquad i, j, l, m = x, y, z \tag{1}$$

where a fixed rectangular Cartesian coordinate system (x, y, z) is used and Einstein's summation convention about repeated indices is understood. The elastic compliance C_{ijlm} in Eq. (1) exhibits the proper conditions of symmetry, and $\bar{\varepsilon}_{ij}$ denotes the eigenstrains. The structure is assumed to be loaded neither by external body forces nor by imposed surface traction, and, hence, no such loads contribute to Eq. (1). This fact is indicated by the index ε, which then refers to the strains $\varepsilon_{ij(\varepsilon)}$ and stresses $\sigma_{lm(\varepsilon)}$ that are solely caused by the eigenstrains $\bar{\varepsilon}_{ij}$. In the following $\bar{\varepsilon}_{ij}$ is not attributed to any specific type of source. From a technological point of view, however, the results presented as such will be of special interest only in those cases where eigenstrains can be produced in the structure in a controlled manner.

The exact meaning of the stress $\sigma_{lm(\varepsilon)}$ in Eq. (1) deserves a clarification. By adopting the German terminology "Eigenspannungen und Eigenspannungsquellen" introduced in his classical paper by Reißner,[22] Mura[1] used the translation "eigenstress" but explicitly reserved this name to self-equilibrating stresses that are caused by eigenstrains in bodies that are free from any external load or surface constraint. Contrary, having in mind structural applications and shape control, bodies of finite dimension and with geometric surface constraints are considered subsequently. The geometric boundary conditions are assumed to be homogeneous because the case of an imposed surface deformation can be treated as a separate load case. The present notation $\sigma_{lm(\varepsilon)}$ therefore refers to the total stresses caused by $\bar{\varepsilon}_{ij}$ in a body, that is, free of imposed external forces but can be subjected to external reaction forces related to the surface constraints. The name "self-stress" has been used by the present authors in some of their previous papers in order to denote this less restricted meaning of $\sigma_{lm(\varepsilon)}$; see, e.g., Ref. 2. In Eq. (1), correspondingly, $\varepsilon_{ij(\varepsilon)}$ denotes the total strain. This strain is related to the corresponding deformations by the linearized relations

$$\varepsilon_{ij(\varepsilon)} = \tfrac{1}{2}[u_{i(\varepsilon),j} + u_{j(\varepsilon),i}] \tag{2}$$

where $u_{i(\varepsilon)}$ is the ith component of the displacement vector of the deformations that are caused by $\bar{\varepsilon}_{ij}$. From Eq. (2) the equations of compatibility are derived by proper differentiation and by eliminating $u_{i(\varepsilon)}$. Therefore, $\varepsilon_{ij(\varepsilon)}$ is called a compatible field of strain.[1] Analogously, a field of eigenstrain $\bar{\varepsilon}_{ij}$ will be called compatible in the following if it can be derived from a field of deformations in the manner of Eq. (2). In general, of course, $\bar{\varepsilon}_{ij}$ represents an incompatible field by not satisfying the equations of compatibility.

Deformations that are produced by eigenstrains are effectively calculated by means of a properly generalized Maysel's formula (see Ziegler and Irschik[23] for a state-of-the-art review). Consider an auxiliary problem, namely the structure without eigenstrain but loaded by a proper force system. For instance, $\sigma_{ij(k)}(\xi, \boldsymbol{x})$ denote the stresses in a point ξ caused by a unit single dummy force applied in $\boldsymbol{x}$ and in the direction of a unit vector $\boldsymbol{e}_k$. The actual displacement in that direction as a result of a field of eigenstrain $u_{k(\varepsilon)}(\boldsymbol{x})$ is given by the following volume integral, which is a (virtual) work expression:

$$u_{k(\varepsilon)}(\boldsymbol{x}) = \int_V \sigma_{ij(k)}(\xi, \boldsymbol{x})\bar{\varepsilon}_{ij}(\xi)\,\mathrm{dV}(\xi) \tag{3}$$

The unit vector $\boldsymbol{e}_k$ may point in any direction and need not to coincide with one of the basis vectors of the Cartesian reference system. This formula has been originally derived by Maysel for the case of isotropic thermal expansion and, using the principle of virtual work, has been extended in Ref. 23 to general eigenstrain analysis. All of the boundary constraints of the finite structure are properly taken into account in Eq. (3). Equation (3) is different from the Mura–Willis integral for infinitely extending bodies[1] but can be easily shown to coincide in this latter case. The boundary constraints of structures may be relaxed in both the eigenstrain and the auxiliary problem of Maysel's formula at the expense of additional boundary integrals (virtual work expressions) occurring in Eq. (3); cf. Ref. 23. The structural form of Maysel's formula using the proper stress resultants is readily available and so is its specialization for point or axisymmetric problems (see again Ref. 23 and the textbook[24]).

Stress-Free Eigenstrain and Static Shape Control

One rather strong motivation for the present study is the observation of special distributions of eigenstrain that do not cause any stress. These distributions have been called "impotent eigenstrains" by Mura,[1] and they are indeed related to the conditions of compatibility for the strains $\varepsilon_{ij(\varepsilon)}$ that are derived by differentiating Eq. (2): it is known that compatible eigenstrains when distributed in a free body do not cause any eigenstress; see again Mura[1] and also Parkus,[25] the latter for the special case of thermal expansion strains. Mura[1] studied an arbitrarily shaped finite inclusion in an infinitely extending anisotropic material, i.e., analogously, he considered a distribution of eigenstrain properly prescribed in the domain of the inclusion, and found that there is no stress field generated in the material if the eigenstrains form a compatible distribution that results from a deformation field vanishing at the surface of the inclusion.

Motivated by these classical results, it is within the scope of the present section to give a systematic classification of stress-free eigenstrains in structures of finite extent and with surface constraints prescribed. Inspection of Eq. (1) reveals that the stress vanishes throughout the structure $\sigma_{lm(\varepsilon)} = 0$ if the distribution of eigenstrain is identical to the one of the total strain $\varepsilon_{ij(\varepsilon)}$, i.e., that it is produced by one and the same $\bar{\varepsilon}_{ij}$. Hence,

$$\bar{\varepsilon}_{ij} = \varepsilon_{ij(\varepsilon)} \tag{4}$$

must hold everywhere in the body. The question arises whether it is possible for Eq. (4) to hold in a structure of finite extent with surface constraints, and this question may be suspected to lead to a nasty inverse mathematical problem for the class of distributions of $\bar{\varepsilon}_{ij}$.

To overcome this problem, we discuss a special, but practically important, class of strains as candidates for stress-free fields of eigenstrain: compatible strains $\varepsilon_{ij(f)}$ are considered that are generated by imposed external forces (f) when acting upon the eigenstrain-free finite structure with surface constraints. In the absence of eigenstrains, the constitutive equations for an anisotropic body subjected to imposed forces or given boundary deformations read as follows:

$$\varepsilon_{ij(f)} = C_{ijlm}\sigma_{lm(f)}, \qquad i, j, l, m = x, y, z \tag{5}$$

The strains are connected to the deformations by the linearized geometric relations

$$\varepsilon_{ij(f)} = \tfrac{1}{2}[u_{i(f),j} + u_{j(f),i}] \tag{6}$$

An integral representation analogous to Maysel's formula (3) can be derived for the force-induced deformations. Consider a virtual deformation applied to the auxiliary equilibrium state of the structure when subjected to the single unit dummy force as used in Eq. (3). According to the principle of virtual work, Ziegler,[24] the total virtual work, done by the external and by the internal forces, has to vanish:

$$\delta W^e_{(k)} + \delta W^i_{(k)} = 0 \tag{7}$$

where the index (k) refers to the auxiliary equilibrium state. Special and admissible virtual deformations are those produced by the system of distributed forces (f), namely $[u_{i(f)}]$ as already introduced. The virtual work of the dummy force done on these deformations becomes

$$\delta W^e_{(k)} = 1\delta u_k(\boldsymbol{x}) = 1 u_{k(f)}(\boldsymbol{x}) \tag{8}$$

whereas the virtual work of the dummy stresses is given by the contracted tensor product

$$\delta W^i_{(k)} = -\int_V \sigma_{ij(k)}(\xi, \boldsymbol{x})\delta\varepsilon_{ij}(\xi)\,\mathrm{d}V(\xi)$$
$$= -\int_V \sigma_{ij(k)}(\xi, \boldsymbol{x})\varepsilon_{ij(f)}(\xi)\,\mathrm{d}V(\xi) \tag{9}$$

The principle of virtual work, Eq. (7), renders upon substitution of expressions (8) and (9)

$$u_{k(f)}(\boldsymbol{x}) = \int_V \sigma_{ij(k)}(\xi, \boldsymbol{x})\varepsilon_{ij(f)}(\xi)\,\mathrm{d}V(\xi) \tag{10}$$

where the boundary conditions are automatically satisfied. The virtual work statement of Eq. (10) directly complements Maysel's formula, Eq. (3). Because the kernel functions of the integrals in Eqs. (3) and (10) are the same, it is immediately seen that the deformations produced by the eigenstrains and those produced by the forces become identical

$$u_{k(\varepsilon)}(\boldsymbol{x}) = u_{k(f)}(\boldsymbol{x}) \tag{11}$$

if the eigenstrain is selected such that

$$\bar{\varepsilon}_{ij}(\boldsymbol{x}) = \varepsilon_{ij(f)}(\boldsymbol{x}) \tag{12}$$

i.e., if one succeeds in applying a distribution of eigenstrain that is equal to the strains produced by imposed forces (f). Because the deformations in that case become equal [Eq. (11)], the kinematic relations, Eqs. (2) and (6), yield the strains produced by the special eigenstrain distribution of Eq. (12) equal to the strains produced by the distributed forces (f):

$$\varepsilon_{ij(\varepsilon)}(\boldsymbol{x}) = \varepsilon_{ij(f)}(\boldsymbol{x}) \tag{13}$$

Substituting Eqs. (12) and (13) into the constitutive equations of the structure in the presence of eigenstrain, it follows that also the stresses vanish:

$$\sigma_{lm(\varepsilon)} = 0 \tag{14}$$

see again the discussion just given with respect to Eq. (4). The eigenstrain distribution suggested in Eq. (12) indeed reproduces its strains, Eqs. (12) and (13), and hence it remains stress free. These eigenstrains form a compatible field, because compatibility holds for the total strains $\varepsilon_{ij(\varepsilon)}$.

From a theoretical point of view, Eqs. (3) and (10) are interpreted to connect the complementary problems of structures either actuated upon by forces or by eigenstrains. As a practical consequence of these relations, both refer to the deformation, that duality can be applied directly to the problem of static shape control. If the right-hand side of Eq. (4) is assigned a negative sign, i.e., when considering Eq. (13), $[-\varepsilon_{ij(f)}]$ is applied as the proper distribution of eigenstrain, then the strains and displacements caused by the imposed forces (f) are exactly annihilated [see Eqs. (12) and (13)]. For the deflections of a beam actuated by piezoelectric strain, this has been utilized by Irschik, et al.[5,6] Because of constraints with respect to the input power, it may not be possible to produce the right amount of eigenstrain required for a complete annihilation in some practical applications; see, e.g., Ref. 26. Nevertheless, the present relation represents an exact benchmark solution for the inverse problem of shape control.

Nonuniqueness of the Shape Control Problem and Deformation-Free Eigenstrain

In the course of derivation of the generalized Maysel's formula, the following orthogonality relation between auxiliary strains caused by dummy forces and the stresses caused by eigenstrain has been derived by Ziegler and Irschik[23]:

$$\int_V \sigma_{ij(\varepsilon)}(\xi)\varepsilon_{ij(k)}(\xi, \boldsymbol{x})\,\mathrm{d}V(\xi) = 0 \tag{15}$$

Substituting Hooke's law for the auxiliary problem gives

$$\int_V \sigma_{ij(\varepsilon)}(\xi)C_{ijlm}(\xi)\sigma_{lm(k)}(\xi, \boldsymbol{x})\,\mathrm{d}V(\xi) = 0 \tag{16}$$

Taking into account the symmetry of the elastic compliance C_{ijlm} and inserting Hooke's law in the presence of eigenstrain [Eq. (1)], one obtains

$$\int_V \sigma_{ij(k)}(\xi, \boldsymbol{x})\bar{\varepsilon}_{ij}(\xi)\,\mathrm{d}V(\xi) = \int_V \sigma_{ij(k)}(\xi, \boldsymbol{x})\varepsilon_{ij(\varepsilon)}(\xi)\,\mathrm{d}V(\xi) \tag{17}$$

Maysel's formula, in generalized form of Eq. (3), is reformulated by substituting Eq. (17):

$$u_{k(\varepsilon)}(\boldsymbol{x}) = \int_V \sigma_{ij(k)}(\xi, \boldsymbol{x})\varepsilon_{ij(\varepsilon)}(\xi)\,\mathrm{d}V(\xi) \tag{18}$$

It is therefore seen that an incompatible field of eigenstrain $\bar{\varepsilon}_{ij}$ and the corresponding strains $\varepsilon_{ij(\varepsilon)}$, when applied as a new field of eigenstrain,

$$\bar{\varepsilon}^*_{ij} = \varepsilon_{ij(\varepsilon)} \tag{19}$$

yield identical deformations. By inspection of the original and the reformulated versions of Maysel's formula, Eqs. (3) and (18) follows

$$u_{k(\varepsilon)} = u^*_{k(\varepsilon)} \tag{20}$$

The superscript $(^*)$ refers to $\bar{\varepsilon}^*_{ij}$. Furthermore, $\bar{\varepsilon}^*_{ij} = \varepsilon_{ij(\varepsilon)}$ represents a stress-free field of eigenstrain. Substitution of Eq. (19) in Eq. (1) consequently yields

$$\sigma^*_{ij(\varepsilon)} = 0 \tag{21}$$

Considering Eq. (20), it is seen that $\bar{\varepsilon}^*_{ij}$ of Eq. (19) represents an exact solution of the shape control of structures subjected to eigenstrain load. However, by definition of a second field of eigenstrain in the form

$$\bar{\varepsilon}^{**}_{ij} = \bar{\varepsilon}_{ij} - \bar{\varepsilon}^*_{ij} = \bar{\varepsilon}_{ij} - \varepsilon_{ij(\varepsilon)} \tag{22}$$

it follows from Eq. (20) that the corresponding deformation vanishes:

$$u^{**}_{k(\varepsilon)} = u_{k(\varepsilon)} - u^*_{k(\varepsilon)} = 0 \tag{23}$$

while the stress still corresponds to the load of eigenstrain $\bar{\varepsilon}_{ij}$:

$$\sigma^{**}_{ij(\varepsilon)} = \sigma_{ij(\varepsilon)} - \sigma^*_{ij(\varepsilon)} = \sigma_{ij(\varepsilon)} \tag{24}$$

Equation (21) is taken into account to identify this relation. From Eq. (23) it is understood that eigenstrains $\bar{\varepsilon}^{**}_{ij}$ of the type represented by Eq. (22) can be added to the exact solution of the static shape control problem, Eqs. (12) and (19), without affecting the required properties. Hence, the static shape control problem is not unique.

Nonuniqueness can be contributed to the fact that static shape control problems represent inverse problems, mathematically defined by an integral equation of the first kind in the form of generalized Maysel's formula, Eq. (3) for the eigenstrain, to be distributed in the structure such that the deformation corresponds to a desired one. Inverse problems often turn out to be ill-posed in the sense of Hadamard. For a survey on mathematical features concerning ill-posed inverse problems in general and for mathematical techniques for overcoming this difficulty in a numerical solution, see Engl.[27] Indeed, it has already been shown that the static shape control is nonunique, which represents one form of an ill-posed problem. It is emphasized that the nonuniqueness of the present inverse problem basically does not influence the practical applicability of the specific solutions derived for these problems. It will be of practical interest for the designer, however, to know about distributions of eigenstrain that can be added to the exact solutions given in Eqs. (12) and (19) without yielding any influence upon the desired effect. Such deformation-free solutions, presented in Eq. (22), which are responsible for nonuniqueness, will be called "nilpotent solutions" of the shape control problem[12] for the case of flexural vibrations of piezoelectric beams. There are various reasons for the designer to be interested in nilpotent solutions: first, an intuitive design may suggest a nilpotent solution of the inverse problem. Such a solution, however, is not suitable for achieving desired deformations of whatever type. Second, in practice, there may be constraints imposed upon the amount of activity present in an actuator problem. By adding nilpotent solutions, constraints may be satisfied, or stress may be diminished [see Eq. (24)]. Moreover, in the case where the desired solution is given by a set of numerical data, e.g., by a measured set, and the solution of the static shape control has to be derived by a numerical procedure it becomes crucial to take into account nonuniqueness by proper regularization techniques in order to obtain numerically stable solutions (see again Engl[27]).

Figure 1 illustrates a redundant truss composed of six member bars of lengths

$$l_1 = l_3 = l_4 = l_5 = \ell, \qquad l_2 = l_6 = \ell\sqrt{2} \tag{25}$$

Hooke's law in absence of eigenstrain is formulated analogous to Eq. (5), but at the level of the theory of idealized trusses

$$\varepsilon_{n(f)} = C_n S_{n(f)}, \qquad n = 1, \dots, 6 \tag{26}$$

where $C_n = (YA)_n^{-1}$ is the cross-sectional compliance of the member numbered n, with Y denoting Young's modulus and A its cross-sectional area. $S_{n(f)}$ denotes the normal force in the nth bar. To represent a sufficiently simple model of an anisotropic body, the tensile compliance of the members are assumed to be nonuniformly distributed, thus putting $C_6 = \alpha C$ being different from $C_1 = C_2 = C_3 = C_4 = C_5 = C$. The truss is assumed to be loaded by a self-equilibrating system of four external forces of equal amount F (Fig. 1). The internal force $S_{6(f)}$ is selected as the redundant one. Nodal equilibrium conditions render the normal forces to be

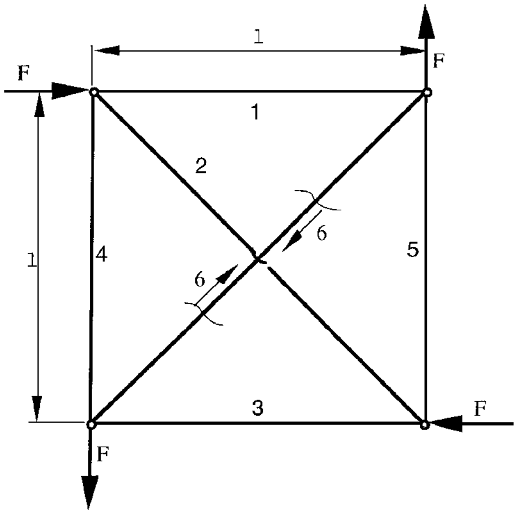

Fig. 1 Shape control of an anisotropic redundant truss.

$$S_{1(f)} = S_{3(f)} = -S_{6(f)}/\sqrt{2}, \qquad S_{2(f)} = -F\sqrt{2} + S_{6(f)}$$
$$S_{4(f)} = S_{5(f)} = F - S_{6(f)}/\sqrt{2} \tag{27}$$

The unknown $S_{6(f)}$ is calculated by means of Menabrea's theorem,[24] i.e., by equating to zero the properly differentiated complementary strain energy $U^c_{(f)}$,

$$\frac{\partial U^c_{(f)}}{\partial S_{6(f)}} = \frac{1}{2}\frac{\partial}{\partial S_{6(f)}}\sum_{n=1}^{6} S^2_{n(f)} C_n l_n = 0 \tag{28}$$

The result is

$$S_{6(f)} = \beta F \tag{29}$$

with the coefficient reflecting the geometry and the anisotropy

$$\beta = \left[1 + \alpha/\left(1+\sqrt{2}\right)\right]^{-1} \tag{30}$$

Strains in the member bars, generated by the system of forces of amount F, thus become

$$\varepsilon_{1(f)} = \varepsilon_{3(f)} = -\beta CF/\sqrt{2}, \qquad \varepsilon_{4(f)} = \varepsilon_{5(f)} = CF\left[1 - \left(\beta/\sqrt{2}\right)\right]$$
$$\varepsilon_{2(f)} = CF\left(\beta - \sqrt{2}\right), \qquad \varepsilon_{6(f)} = \alpha\beta CF \tag{31}$$

Following the suggestion explicitly contained in Eq. (12), the following candidates for a stress-free eigenstrain distribution are consequently studied:

$$\bar{\varepsilon}_n = \varepsilon_{n(f)}, \qquad n = 1, \dots, 6 \tag{32}$$

with $\varepsilon_{n(f)}$ of Eq. (31). The corresponding internal forces are also proportional to the redundant normal force $S_{6(\varepsilon)}$; Eqs. (27) and (29) when combined render

$$S_{1(\varepsilon)} = S_{3(\varepsilon)} = S_{4(\varepsilon)} = S_{5(\varepsilon)} = -S_{6(\varepsilon)}/\sqrt{2}, \qquad S_{2(\varepsilon)} = S_{6(\varepsilon)} \tag{33}$$

In the presence of eigenstrain, the complementary strain energy includes additional terms[24]:

$$U^c_{(\varepsilon)} = \sum_{n=1}^{6}\left[\tfrac{1}{2}C_n S^2_{n(\varepsilon)} + S_{n(\varepsilon)}\bar{\varepsilon}_n\right] l_n \tag{34}$$

Castigliano's theorem in the form of Menabrea requires

$$\frac{\partial U^*_{(\varepsilon)}}{\partial S_{6(\varepsilon)}} = \sum_{n=1}^{6}\frac{\partial S_{n(\varepsilon)}}{\partial S_{6(\varepsilon)}}\left[C_n S_{n(\varepsilon)} + \bar{\varepsilon}_n\right] l_n = 0 \tag{35}$$

Substituting Eqs. (31–33) into Eq. (35) yields

$$-\left(1/\sqrt{2}\right)\left[-4C\left(S_{6(\varepsilon)}/\sqrt{2}\right) - 2\left(CF\beta/\sqrt{2}\right)\right.$$
$$\left. + 2CF\left(1 - \beta/\sqrt{2}\right)\right]\ell + \left[CS_{6(\varepsilon)} + CF\left(\beta - \sqrt{2}\right)\right]\ell\sqrt{2}$$
$$+ \left[\alpha CS_{6(\varepsilon)} + \alpha CF\beta\right]\ell\sqrt{2} = 0 \tag{36}$$

Considering the value of the coefficient β [Eq. (30)], the solution of Eq. (36) for the unknown $S_{6(\varepsilon)}$ turns out to be the trivial one. Hence, from Eq. (33)

$$S_{6(\varepsilon)} = S_{j(\varepsilon)} = 0, \qquad j = 1, \dots, 5 \tag{37}$$

It has thus been demonstrated that the stresses in the statically indeterminate truss which are caused by the nonuniform eigenstrain distribution of Eq. (32) do vanish, as has been predicted by Eq. (14). Putting Eq. (37) into Hooke's law of the member rods in the presence of eigenstrain

$$\varepsilon_{n(\varepsilon)} - \bar{\varepsilon}_n = C_n S_{n(\varepsilon)}, \qquad n = 1, \dots, 6 \tag{38}$$

leads to

$$\varepsilon_{n(\varepsilon)} = \bar{\varepsilon}_n = \varepsilon_{n(f)}, \qquad n = 1, \dots, 6 \tag{39}$$

i.e., strains caused by the eigenstrain load under consideration, and the strains caused by the given force loading are identical. Thus, the nonuniform eigenstrain distribution defined by Eqs. (30) and (31) is stress-free, and it is an exact solution of the shape control problem; also refer to the three-dimensional results of Eqs. (13) and (14).

Next, an incompatible distribution of eigenstrain is studied, where only the bar numbered 6 is assumed to be actuated:

$$\bar{\varepsilon}_j = 0, \qquad j = 1, \ldots, 5, \qquad \bar{\varepsilon}_6 = \bar{\varepsilon} \tag{40}$$

Castigliano's principle [Eq. (35)] renders in this case

$$S_{6(\varepsilon)} = -\left[\bar{\varepsilon}/C\left(1 + \sqrt{2} + \alpha\right)\right] \tag{41}$$

The total strains thus become

$$\varepsilon_{1(\varepsilon)} = \varepsilon_{3(\varepsilon)} = \varepsilon_{4(\varepsilon)} = \varepsilon_{5(\varepsilon)} = -CS_{6(\varepsilon)}/\sqrt{2}$$

$$\varepsilon_{2(\varepsilon)} = CS_{6(\varepsilon)}, \qquad \varepsilon_{6(\varepsilon)} = \alpha CS_{6(\varepsilon)} + \bar{\varepsilon} = \beta\bar{\varepsilon} \tag{42}$$

According to Eq. (19), the following nonuniform candidate for a stress-free eigenstrain distribution is studied subsequently:

$$\bar{\varepsilon}^*_n = \varepsilon_{n(\varepsilon)}, \qquad n = 1, \ldots, 6 \tag{43}$$

with $\varepsilon_{n(\varepsilon)}$ of Eq. (42). Castigliano's principle [Eq. (35)] then gives

$$-\left(\ell/\sqrt{2}\right)\left[-4C\left(S^*_{6(\varepsilon)}/\sqrt{2}\right) + 4\bar{\varepsilon}/\sqrt{2}\left(1 + \sqrt{2} + \alpha\right)\right] + \ell\sqrt{2}\left[CS^*_{6(\varepsilon)} - \bar{\varepsilon}/\left(1 + \sqrt{2} + \alpha\right) + \alpha CS^*_{6(\varepsilon)} + \beta\bar{\varepsilon}\right] = 0 \tag{44}$$

Hence, after substituting Eq. (30) it is seen that the redundant force does vanish for the eigenstrains of Eq. (43):

$$S^*_{6(\varepsilon)} = 0 \tag{45}$$

From Eq. (33) the other internal forces then vanish, too:

$$S^*_{j(\varepsilon)} = 0, \qquad j = 1, \ldots, 5 \tag{46}$$

Thus, also the distribution of eigenstrain according to Eqs. (43) and (44) is a stress-free one. From Hooke's law in the presence of eigenstrain [Eq. (38)] and with $S^*_{n(\varepsilon)}$ of Eqs. (45) and (46), it is finally proved that

$$\varepsilon^*_{n(\varepsilon)} = \bar{\varepsilon}^*_n = \varepsilon_{n(\varepsilon)}, \qquad n = 1, \ldots, 6 \tag{47}$$

Note that at the level of the redundant truss of Fig. 1, Eqs. (45–47) render evidence of the validity of the three-dimensional results presented in Eqs. (20) and (21). This remark completes the presented example.

Eigenstrain distributions in Eqs. (32), (40), and (43) may be implemented in practice by selecting the rods of the truss from stackers made of smart materials, which may also be used for the purpose of sensing the deformations (see Tzou[28] for a recent review). Some applications are collected in Refs. 29 and 30.

It has been demonstrated that the static shape control of structures represents an ill-posed problem. A class of deformation-free fields of eigenstrain has been identified causing the nonuniqueness. This class is constituted by the field of compatible strains, which is caused by any field of eigenstrain, both compatible ones and incompatible ones. The more specialized theorem of linear elasticity should be mentioned here so that if any part of the boundary, however small, is free of traction and the displacements vanish in the same region then the enclosed body is free of stresses. It was proved by Almansi in 1907 (see Ref. 31 for the plane problem).

Conclusions

Having identified deformation-free fields of eigenstrain, an exact solution for the static shape control problem by means of eigenstrain actuation has been given. This exact solution is provided for force-loaded linear elastic anisotropic structures by using the generalized formula (of thermoelasticity) attributed to Maysel. If it is possible (by means of either piezoelectric or thermal sources) to produce eigenstrains equal to the compatible field of the load strains, these eigenstrains render the same deformations (displacements) as the external force load do. Further, it has been shown that such a field of eigenstrains does not produce any stress.

An anisotropic redundant truss serves as an illustrative example for deformation control by eigenstrain. Thus, the optimal control of a discretized system given a limited proof of the optimal solution of the three-dimensional continuum is obtained. The novel, easy to apply, solution technique, based on the generalized Maysel formula, depends on the interpretation of either piezoelectrically or thermally produced strains as eigenstrain in the sense of Mura's.

For aeronautical, biomechanical, and general industrial applications, two practicable possibilities for realizing deformation control by eigenstrains exist. One relies on the choice of smart materials with distributed electrical actuators and the second on the availability of functionally graded materials subjected to controlled temperature fields.

Acknowledgments

Support of Hans Irschik by a grant of the Austrian National Science Fund (FWF), Contract P11993-TEC, is gratefully acknowledged. Franz Ziegler acknowledges the INTAS-ESA Research Project 99-0185, where he functions as one of the four principal investigators and as the coordinator.

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A. M. Baz
Associate Editor